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Viscosity approximation methods for hierarchical optimization problems in CAT(0) spaces

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Abstract
This paper aims at investigating viscosity approximation methods for solving a system of variational inequalities in a CAT(0) space. Two algorithms are given. Under certain appropriate conditions, we prove that the iterative schemes converge strongly to the unique solution of the hierarchical optimization problem. The result presented in this paper mainly improves and extends the corresponding results of Shi and Chen (J. Appl. Math. 2012:421050, 2012, doi:10.1155/2012/421050), Wangkeeree and Preechasilp (J. Inequal. Appl. 2013:93, 2013, doi:10.1186/1029-242X-2013-93) and others.

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1 Introduction

The concept of variational inequalities plays an important role in various kinds of problems in pure and applied sciences (see, for example, [1–11]). Moreover, the rapid development and the prolific growth of the theory of variational inequalities have been made by many researchers.

In a CAT(0) space, Saejung [12] studied the convergence theorems of the following Halpern iterations for a nonexpansive mapping T. Let u be fixed and $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tu \oplus (1-t)Tx$; *i.e.*,

$$x_t = tu \oplus (1-t)Tx_t, \tag{1.1}$$

where $t \in [0,1]$ and $x_0, u \in C$ are arbitrarily chosen and

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

where $\alpha_n \in (0,1)$. It is proved that $\{x_t\}$ converges strongly as $t \to 0$ to $\tilde{x} \in F(T)$ such that $\tilde{x} = P_{F(T)}u$, and $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x} \in F(T)$ under certain appropriate conditions on α_n, where $P_C x$ is a metric projection from X onto C.

In 2012, Shi and Chen [13] studied the convergence theorems of the following Moudafi viscosity iterations for a nonexpansive mapping T: For a contraction f on C and $t \in (0,1)$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$; *i.e.*,

$$x_t = tf(x_t) \oplus (1-t)Tx_t, \tag{1.3}$$

and $x_0 \in C$ is arbitrarily chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.4}$$

where $\{\alpha_n\} \subset (0,1)$. They proved that $\{x_t\}$ defined by (1.3) converges strongly as $t \to 0$ to $\tilde{x} \in F(T)$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ in the framework of a CAT(0) space satisfying the property $\mathscr{P}$, *i.e.*, if for $x, u, y_1, y_2 \in X$,

$$d(x, P_{[x,y_1]}u)d(x,y_1) \leq d(x, P_{[x,y_2]}u)d(x,y_2) + d(x,u)d(y_1,y_2).$$

Furthermore, they also obtained that $\{x_n\}$ defined by (1.4) converges strongly as $n \to \infty$ to $\tilde{x} \in F(T)$ under certain appropriate conditions imposed on $\{\alpha_n\}$.

By using the concept of *quasilinearization*, which was introduced by Berg and Nikolaev [14], Wangkeeree and Preechasilp [15] studied the strong convergence theorems of iterative schemes (1.3) and (1.4) in CAT(0) spaces without the property $\mathscr{P}$. They proved that iterative schemes (1.3) and (1.4) converge strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$, which is the unique solution of the variational inequality (VIP)

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{1.5}$$

In this paper, we are interested in the following so-called *hierarchical optimization problems* (HOP). More precisely, let $f, g : C \to C$ be two contractions with coefficient $\alpha \in (0,1)$, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings such that $F(T_1)$ and $F(T_2)$ are nonempty. The class of *hierarchical optimization problems* (HOP) consists in finding $(\tilde{x}, \tilde{y}) \in F(T_1) \times F(T_2)$ such that the following inequalities hold:

$$\begin{cases} \langle \overrightarrow{\tilde{x}f(\tilde{y})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, & \forall x \in F(T_1), \\ \langle \overrightarrow{\tilde{y}g(\tilde{x})}, \overrightarrow{y\tilde{y}} \rangle \geq 0, & \forall y \in F(T_2). \end{cases} \tag{1.6}$$

For this purpose, we introduce the following iterative schemes:

$$\begin{cases} x_t = tf(T_2 y_t) \oplus (1-t)T_1 x_t, \\ y_t = tg(T_1 x_t) \oplus (1-t)T_2 y_t, \end{cases} \tag{1.7}$$

where $t \in (0,1)$, and

$$\begin{cases} x_0, y_0 \in C, \\ x_{n+1} = \alpha_n f(T_2 y_n) \oplus (1-\alpha_n)T_1 x_n, \\ y_{n+1} = \alpha_n g(T_1 x_n) \oplus (1-\alpha_n)T_2 x_n, \quad n \geq 0, \end{cases} \tag{1.8}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies

(H1) $\alpha_n \to 0$,

(H2) $\sum_{n=0}^{\infty} = \infty$,
(H3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

We prove that iterative schemes (1.7) and (1.8) converge strongly to $(\tilde{x}, \tilde{y}) \in F(T_1) \times F(T_2)$ such that $\tilde{x} = P_{F(T_1)} f(\tilde{y})$ and $\tilde{y} = P_{F(T_2)} g(\tilde{x})$, which is the unique solution of (1.6).

2 Preliminaries

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in Y$ (or, more briefly, a geodesic from x to y) is a map $c : [0, l] \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a *geodesic segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, and x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in 1, 2, 3$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \le d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

Let $x, y \in X$ by [16, Lemma 2.1(iv)] for each $t \in [0, 1]$, then there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \qquad d(y, z) = (1 - t)d(x, y). \tag{2.1}$$

From now on, we will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.1).

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.1 *Let X be a* CAT(0) *space. Then*

(i) (*see* [16, Lemma 2.4]) *for each $x, y, z \in X$ and $t \in [0, 1]$, one has*

$$d\big((1 - t)x \oplus ty, z\big) \le (1 - t)d(x, z) + td(y, z); \tag{2.2}$$

(ii) (*see* [17]) *for each $x, y, z \in X$ and $t, s \in [0, 1]$, one has*

$$d\big((1 - t)x \oplus ty, (1 - s)x \oplus sy\big) \le |t - s|d(x, y); \tag{2.3}$$

(iii) (*see* [18]) *for each $x, y, z, w \in X$ and $t \in [0, 1]$, one has*

$$d\big((1 - t)x \oplus ty, (1 - t)z \oplus tw\big) \le (1 - t)d(x, z) + td(y, w); \tag{2.4}$$

(iv) (*see* [19]) *for each* $x, y, z \in X$ *and* $t \in [0, 1]$, *one has*

$$d\big((1-t)z \oplus tx, (1-t)z \oplus ty\big) \le td(x, y); \tag{2.5}$$

(v) (*see* [16]) *for each* $x, y, z \in X$ *and* $t \in [0, 1]$, *one has*

$$d^2\big((1-t)x \oplus ty, z\big) \le (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y). \tag{2.6}$$

Let C be a nonempty subset of a complete CAT(0) space X. Recall that a self-mapping $T : C \to C$ is a nonexpansion on C iff $d(Tx, Ty) \le d(x, y)$ for all $x, y \in C$. A point $x \in C$ is called a fixed point of T if $x = Tx$. We denote by $F(T)$ the set of all fixed points of T. A self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that $d(fx, fy) \le \alpha d(x, y)$. Banach's contraction principle [20] guarantees that f has a unique fixed point when C is a nonempty closed convex subset of a complete metric space.

Fixed-point theory in CAT(0) spaces was first studied by Kirk (see [19, 21]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed.

Berg and Nikolaev [14] introduced the concept of *quasilinearization* as follows.

Let (X, d) be a metric space. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then *quasilinearization* is defined as a mapping $\langle\cdot,\cdot\rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \frac{1}{2}\big(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\big), \quad a, b, c, d \in X. \tag{2.7}$$

It is easily seen that $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \langle\overrightarrow{cd}, \overrightarrow{ab}\rangle$, $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = -\langle\overrightarrow{ba}, \overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ax}, \overrightarrow{cd}\rangle + \langle\overrightarrow{xb}, \overrightarrow{cd}\rangle = \langle\overrightarrow{ab}, \overrightarrow{cd}\rangle$ for all $a, b, c, d, x \in X$.

We say that X satisfies the *Cauchy-Schwarz inequality* if

$$\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle \le d(a, b)d(c, d) \tag{2.8}$$

for all $a, b, c, d \in X$.

It is known [14, Corollary 3] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Recently, Dehghan and Rooin [22] presented a characterization of a metric projection in CAT(0) spaces as follows.

Lemma 2.2 *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if $\langle\overrightarrow{yu}, \overrightarrow{ux}\rangle \ge 0$ for all $y \in C$.*

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [23] that for each bounded sequence $\{x_n\}$ in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to $\triangle$-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. We use $x_n \stackrel{\triangle}{\to} x$ to denote that $\{x_n\}$ $\triangle$-converges to x. The uniqueness of an asymptotic center implies that the CAT(0) space X satisfies Opial's property, *i.e.*, for given $\{x_n\} \subset X$ such that $x_n \stackrel{\triangle}{\to} x$, then for any given $y \in X$ with $y \neq x$, the following holds:

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Lemma 2.3 [24] *Assume that X is a complete* CAT(0) *space. Then*:
(i) *Every bounded sequence in X always has a $\triangle$-convergent subsequence.*
(ii) *If C is a closed convex subset of X and $T : C \to X$ is a nonexpansive mapping, then the conditions $x_n \stackrel{\triangle}{\to} x$ and $d(x_n, Tx_n) \to 0$ imply $x \in C$ and $Tx = x$.*

The following lemma shows a characterization of $\triangle$-convergence.

Lemma 2.4 [23] *Let X be a complete* CAT(0) *space, $\{x_n\}$ be a sequence in X, and $x \in X$. Then $x_n \stackrel{\triangle}{\to} x$ if and only if* $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \le 0$ *for all $y \in X$.*

Lemma 2.5 [23] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\beta_n, \quad n \ge 0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset \mathbb{R}$ such that
1. $\sum_{n=0}^{\infty} \alpha_n = \infty$;
2. $\limsup_{n\to\infty} \beta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\{a_n\}$ converges to zero as $n \to \infty$.

Lemma 2.6 [15] *Let X be a complete* CAT(0) *space. Then,*
(i) *for each $u, x, y \in X$, one has*

$$d^2(x, u) \le d^2(y, u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle; \tag{2.9}$$

(ii) *for any $u, v \in X$ and $t \in [0,1]$, letting $u_t = tu \oplus (1 - t)v$ for all $x, y \in X$, we have*:
(a) $\langle \overrightarrow{u_tx}, \overrightarrow{u_ty} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{u_ty} \rangle + (1 - t)\langle \overrightarrow{vx}, \overrightarrow{u_ty} \rangle$;
(b) $\langle \overrightarrow{u_tx}, \overrightarrow{uy} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1 - t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$;
(c) $\langle \overrightarrow{u_tx}, \overrightarrow{u_ty} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1 - t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

3 Main results

Now we are ready to give our main results in this paper.

Let (X, d) be a metric space. Define a mapping $\hat{d} : (X \times X) \times (X \times X) \to \mathbb{R}^+$ by

$$\hat{d}((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$$

for all $x_1, x_2, y_1, y_2 \in X$. Then it is easy to verify that $(X \times X, \hat{d})$ is a metric space, and $(X \times X, \hat{d})$ is complete if and only if (X, d) is complete.

Lemma 3.1 *Let C be a closed convex subset of a complete* CAT(0) *space. Let $f, g : C \to C$ be two contractions with coefficient $\alpha \in (0,1)$, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings. For any $t \in (0,1)$, define another mapping $G_t : C \times C \to C \times C$ by*

$$G_t(x,y) = \big(tf(T_2y) \oplus (1-t)T_1x, tg(T_1x) \oplus (1-t)T_2y\big).$$

Then G_t is a contraction on $C \times C$.

Proof For any $(x_1,y_1), (x_2,y_2) \in C \times C$ and $t \in (0,1)$, we have

$$\begin{aligned}
\hat{d}\big(G_t(x_1,y_1), G_t(x_2,y_2)\big) &= d\big(tf(T_2y_1) \oplus (1-t)T_1x_1, tf(T_2y_2) \oplus (1-t)T_1x_2\big) \\
&\quad + d\big(tg(T_1x_1) \oplus (1-t)T_2y_1, tg(T_1x_2) \oplus (1-t)T_2y_2\big) \\
&\le td\big(f(T_2y_1), f(T_2y_2)\big) + (1-t)d(T_1x_1, T_1x_2) \\
&\quad + td\big(g(T_1x_1), g(T_1x_2)\big) + (1-t)d(T_2y_1, T_2y_2) \\
&\le \big(1 - t(1-\alpha)\big)\big(d(x_1,x_2) + d(y_1,y_2)\big) \\
&= \big(1 - t(1-\alpha)\big)\hat{d}\big((x_1,y_1),(x_2,y_2)\big).
\end{aligned}$$

This implies that G_t is a contraction mapping. Therefore there exists a unique fixed point $(x_t, y_t) \in C \times C$ of G_t such that

$$\begin{cases} x_t = tf(T_2y_t) \oplus (1-t)T_1x_t, \\ y_t = tg(T_1x_t) \oplus (1-t)T_2y_t. \end{cases}$$ □

Theorem 3.2 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings such that $F(T_1)$ and $F(T_2)$ are nonempty. Let f, g be two contractions on C with coefficient $0 < \alpha < 1$. For each $t \in (0,1]$, let $\{x_t\}$ and $\{y_t\}$ be given by* (1.7). *Then $x_t \to \tilde{x}$ and $y_t \to \tilde{y}$ as $t \to 0$ such that $\tilde{x} = P_{F(T_1)}f(\tilde{y})$, $\tilde{y} = P_{F(T_2)}g(\tilde{x})$ which is the unique solution of HOP* (1.6).

Proof We first show that $\{x_t\}$ and $\{y_t\}$ are bounded. Indeed, take $(p,q) \in F(T_1) \times F(T_2)$ to derive that

$$\begin{aligned}
d(x_t,p) + d(y_t,q) &= d\big(tf(T_2y_t) \oplus (1-t)T_1x_t, p\big) + d\big(tg(T_1x_t) \oplus (1-t)T_2y_t, q\big) \\
&\le td\big(f(T_2y_t), p\big) + (1-t)d(T_1x_t, p) + td\big(g(T_1x_t), q\big) + (1-t)d(T_2y_t, q) \\
&\le td\big(f(T_2y_t), f(q)\big) + td\big(f(q), p\big) + (1-t)d(T_1x_t, p) \\
&\quad + td\big(g(T_1x_t), g(p)\big) + td\big(g(p), q\big) + (1-t)d(T_2y_t, q) \\
&\le t\alpha d(y_t, q) + td\big(f(q), p\big) + (1-t)d(x_t, p) \\
&\quad + t\alpha d(x_t, p) + td\big(g(p), q\big) + (1-t)d(y_t, q).
\end{aligned}$$

After simplifying, we have

$$d(x_t,p)+d(y_t,q)\le\frac{1}{1-\alpha}\bigl(d\bigl(f(q),p\bigr)+d\bigl(g(p),q\bigr)\bigr).$$

Hence $\{x_t\}$ and $\{y_t\}$ are bounded, so are $\{T_1x_t\}$, $\{T_2y_t\}$, $\{f(T_2y_t)\}$ and $\{g(T_1x_t)\}$. Consequently,

$$\begin{aligned}d(x_t,T_1x_t)+d(y_t,T_2y_t)&=d\bigl(tf(T_2y_t)\oplus(1-t)T_1x_t,T_1x_t\bigr)\\&\quad+d\bigl(tg(T_1x_t)\oplus(1-t)T_2y_t,T_2y_t\bigr)\\&=td\bigl(f(T_2y_t),T_2y_t\bigr)+td\bigl(g(T_1x_t),T_2y_t\bigr)\to0\quad(\text{as }t\to0).\end{aligned}$$

In particular, we have

$$d(x_t,T_1x_t)\to0,\qquad d(y_t,T_2y_t)\to0\quad(\text{as }t\to0).\tag{3.1}$$

Next we prove that $\{x_t\}$ is relatively compact as $t\to0$.

In fact, let $\{t_n\}\subset(0,1)$ be any subsequence such that $t_n\to0$ as $n\to\infty$. Put $x_n:=x_{t_n}$ and $y_n:=y_{t_n}$. Now we prove that $\{(x_n,y_n)\}$ contains a subsequence converging strongly to $(\tilde{x},\tilde{y})$ where $\tilde{x}=P_{F(T_1)}f(\tilde{y})$, $\tilde{y}=P_{F(T_2)}g(\tilde{x})$ and it is a solution of HOP (1.6).

In fact, since $\{x_n\}$ and $\{y_n\}$ are both bounded, by Lemma 2.3(i), (ii) and (3.1), we may assume that $x_n\xrightarrow{\triangle}\tilde{x}$ and $y_n\xrightarrow{\triangle}\tilde{y}$, and $\tilde{x}\in F(T_1)$, $\tilde{y}\in F(T_2)$. Hence it follows from Lemma 2.6 that

$$\begin{aligned}d^2(x_n,\tilde{x})+d^2(y_n,\tilde{y})&=\langle\overrightarrow{x_n\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle+\langle\overrightarrow{y_n\tilde{y}},\overrightarrow{y_n\tilde{y}}\rangle\\&\le t_n\bigl\langle\overrightarrow{f(T_2y_n)\tilde{x}},\overrightarrow{x_n\tilde{x}}\bigr\rangle+(1-t_n)\bigl\langle\overrightarrow{(T_1x_n)\tilde{x}},\overrightarrow{x_n\tilde{x}}\bigr\rangle\\&\quad+t_n\bigl\langle\overrightarrow{g(T_1x_n)\tilde{y}},\overrightarrow{y_n\tilde{y}}\bigr\rangle+(1-t_n)\bigl\langle\overrightarrow{(T_2y_n)\tilde{y}},\overrightarrow{y_n\tilde{y}}\bigr\rangle\\&\le t_n\bigl\langle\overrightarrow{f(T_2y_n)\tilde{x}},\overrightarrow{x_n\tilde{x}}\bigr\rangle+(1-t_n)d(T_1x_n,\tilde{x})d(x_n,\tilde{x})\\&\quad+t_n\bigl\langle\overrightarrow{g(T_1x_n)\tilde{y}},\overrightarrow{y_n\tilde{y}}\bigr\rangle+(1-t_n)d(T_2y_n,\tilde{y})d(y_n,\tilde{y})\\&\le t_n\bigl\langle\overrightarrow{f(T_2y_n)\tilde{x}},\overrightarrow{x_n\tilde{x}}\bigr\rangle+(1-t_n)d^2(x_n,\tilde{x})\\&\quad+t_n\bigl\langle\overrightarrow{g(T_1x_n)\tilde{y}},\overrightarrow{y_n\tilde{y}}\bigr\rangle+(1-t_n)d^2(y_n,\tilde{y}).\end{aligned}\tag{3.2}$$

After simplifying, we have

$$\begin{aligned}d^2(x_n,\tilde{x})+d^2(y_n,\tilde{y})&\le\bigl\langle\overrightarrow{f(T_2y_n)\tilde{x}},\overrightarrow{x_n\tilde{x}}\bigr\rangle+\bigl\langle\overrightarrow{g(T_1x_n)\tilde{y}},\overrightarrow{y_n\tilde{y}}\bigr\rangle\\&=\bigl\langle\overrightarrow{f(T_2y_n)f(\tilde{y})},\overrightarrow{x_n\tilde{x}}\bigr\rangle+\bigl\langle\overrightarrow{f(\tilde{y})\tilde{x}},\overrightarrow{x_n\tilde{x}}\bigr\rangle\\&\quad+\bigl\langle\overrightarrow{g(T_1x_n)g(\tilde{x})},\overrightarrow{y_n\tilde{y}}\bigr\rangle+\bigl\langle\overrightarrow{g(\tilde{x})\tilde{y}},\overrightarrow{y_n\tilde{y}}\bigr\rangle\\&\le d\bigl(f(T_2y_n),f(\tilde{y})\bigr)d(x_n,\tilde{x})+\bigl\langle\overrightarrow{f(\tilde{y})\tilde{x}},\overrightarrow{x_n\tilde{x}}\bigr\rangle\\&\quad+d\bigl(g(T_1x_n),g(\tilde{x})\bigr)d(y_n,\tilde{y})+\bigl\langle\overrightarrow{g(\tilde{x})\tilde{y}},\overrightarrow{y_n\tilde{y}}\bigr\rangle\\&\le2\alpha d(x_n,\tilde{x})d(y_n,\tilde{y})+\bigl\langle\overrightarrow{f(\tilde{y})\tilde{x}},\overrightarrow{x_n\tilde{x}}\bigr\rangle+\bigl\langle\overrightarrow{g(\tilde{x})\tilde{y}},\overrightarrow{y_n\tilde{y}}\bigr\rangle\\&\le\alpha\bigl(d^2(x_n,\tilde{x})+d^2(y_n,\tilde{y})\bigr)+\bigl\langle\overrightarrow{f(\tilde{y})\tilde{x}},\overrightarrow{x_n\tilde{x}}\bigr\rangle+\bigl\langle\overrightarrow{g(\tilde{x})\tilde{y}},\overrightarrow{y_n\tilde{y}}\bigr\rangle\end{aligned}$$

and thus

$$d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y}) \le \frac{1}{1-\alpha}\left[\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n\tilde{y}}\rangle\right]. \tag{3.3}$$

Since $x_n \xrightarrow{\triangle} \tilde{x}$ and $y_n \xrightarrow{\triangle} \tilde{y}$, by Lemma 2.4,

$$\limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle \le 0 \quad \text{and} \quad \limsup_{n\to\infty}\langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n\tilde{y}}\rangle \le 0.$$

Hence we have

$$\limsup_{n\to\infty}\left[\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n\tilde{y}}\rangle\right] \le \limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle + \limsup_{n\to\infty}\langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n\tilde{y}}\rangle \le 0. \tag{3.4}$$

It follows from (3.3) that $d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y}) \to 0$. Hence $x_n \to \tilde{x}$ and $y_n \to \tilde{y}$.

Next we show that $(\tilde{x},\tilde{y}) \in F(T_1) \times F(T_2)$, which solves HOP (1.6).

Indeed, for each $x \in F(T_1)$, $y \in F(T_2)$, we have

$$\begin{aligned} d^2(x_t,x) &= d^2\big(tf(T_2y_t) \oplus (1-t)T_1x_t, x\big) \\ &\le td^2\big(f(T_2y_t),x\big) + (1-t)d^2(T_1x_t,x) - t(1-t)d^2\big(f(T_2y_t),T_1x_t\big) \\ &\le td^2\big(f(T_2y_t),x\big) + (1-t)d^2(x_t,x) - t(1-t)d^2\big(f(T_2y_t),T_1x_t\big). \end{aligned}$$

This implies that

$$d^2(x_t,x) \le d^2\big(f(T_2y_t),x\big) - (1-t)d^2\big(f(T_2y_t),T_1x_t\big).$$

Letting $t = t_n \to 0$ and taking the limit and noting that $d(y_t, T_2y_t) \to 0$ and $d(x_t, T_1x_t) \to 0$, we have

$$d^2(\tilde{x},x) \le d^2\big(f(\tilde{y}),x\big) - d^2\big(f(\tilde{y}),\tilde{x}\big).$$

Hence

$$\langle \overrightarrow{\tilde{x}f(\tilde{y})}, \overrightarrow{x\tilde{x}}\rangle = \frac{1}{2}\left[\tilde{x} + d^2\big(f(\tilde{y}),x\big) - d^2\big(f(\tilde{y}),\tilde{x}\big) - d^2(\tilde{x},x)\right] \ge 0.$$

It is similar to proving that

$$\langle \overrightarrow{\tilde{y}g(\tilde{x})}, \overrightarrow{y\tilde{y}}\rangle \ge 0.$$

That is, $(\tilde{x},\tilde{y})$ solves inequality (1.6).

Finally, we show that the entire net $\{x_t\}$ converges to $\tilde{x}$, and $\{y_t\}$ converges to $\tilde{y}$. In fact, for any subsequence $\{s_n\} \subset (0,1)$ such that $s_n \to 0$ (as $n \to \infty$), assume that $x_{s_n} \to \hat{x}$ and $y_{s_n} \to \hat{y}$. By the same argument as above, we get that $(\hat{x},\hat{y}) \in F(T_1) \times F(T_2)$ and solves inequality (1.6). Hence we have

$$\begin{cases} \langle \overrightarrow{\tilde{x}f(\tilde{y})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \le 0, \\ \langle \overrightarrow{\tilde{y}g(\tilde{x})}, \overrightarrow{\tilde{y}\hat{y}}\rangle \le 0 \end{cases} \tag{3.5}$$

and

$$\begin{cases} \langle \overrightarrow{\hat{x}f(\hat{y})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \leq 0, \\ \langle \overrightarrow{\hat{y}g(\hat{x})}, \overrightarrow{\tilde{y}\hat{y}} \rangle \leq 0. \end{cases} \tag{3.6}$$

Adding up (3.5) and (3.6), we get that

$$\begin{aligned} 0 &\geq \langle \overrightarrow{\tilde{x}f(\tilde{y})}, \overrightarrow{\tilde{x}\hat{x}} \rangle + \langle \overrightarrow{\tilde{y}g(\tilde{x})}, \overrightarrow{\tilde{y}\hat{y}} \rangle - \langle \overrightarrow{\hat{x}f(\hat{y})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\hat{y}g(\hat{x})}, \overrightarrow{\tilde{y}\hat{y}} \rangle \\ &= \langle \overrightarrow{\tilde{x}f(\tilde{y})}, \overrightarrow{\tilde{x}\hat{x}} \rangle + \langle \overrightarrow{f(\hat{y})f(\tilde{y})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\hat{x}\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\tilde{x}f(\tilde{y})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\ &\quad + \langle \overrightarrow{\tilde{y}g(\tilde{x})}, \overrightarrow{\tilde{y}\hat{y}} \rangle + \langle \overrightarrow{g(\hat{x})g(\tilde{x})}, \overrightarrow{\tilde{y}\hat{y}} \rangle - \langle \overrightarrow{\hat{y}\tilde{y}}, \overrightarrow{\tilde{y}\hat{y}} \rangle - \langle \overrightarrow{\tilde{y}g(\tilde{x})}, \overrightarrow{\tilde{y}\hat{y}} \rangle \\ &= \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle + \langle \overrightarrow{\tilde{y}\hat{y}}, \overrightarrow{\tilde{y}\hat{y}} \rangle - \langle \overrightarrow{f(\tilde{y})f(\hat{y})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{g(\tilde{x})g(\hat{x})}, \overrightarrow{\tilde{y}\hat{y}} \rangle \\ &\geq d^2(\tilde{x},\hat{x}) + d^2(\tilde{y},\hat{y}) - d\big(f(\tilde{y}),f(\hat{y})\big)d(\tilde{x},\hat{x}) - d\big(g(\tilde{x}),g(\hat{x})\big)d(\tilde{y},\hat{y}) \\ &\geq d^2(\tilde{x},\hat{x}) + d^2(\tilde{y},\hat{y}) - 2\alpha d(\tilde{y},\hat{y})d(\tilde{x},\hat{x}) \\ &\geq (1-\alpha)\big[d^2(\tilde{x},\hat{x}) + d^2(\tilde{y},\hat{y})\big]. \end{aligned}$$

Since $0 < \alpha < 1$, we have that $d^2(\tilde{x},\hat{x}) + d^2(\tilde{y},\hat{y}) = 0$, and so $\tilde{x} = \hat{x}$, $\tilde{y} = \hat{y}$. Hence the entire net $\{x_t\}$ converges to $\tilde{x}$ and $\{y_t\}$ converges to $\tilde{y}$, which $(\hat{x},\hat{y})$ solves HOP (1.6). This completes the proof of Theorem 3.2. □

Theorem 3.3 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings such that $F(T_1)$ and $F(T_2)$ are nonempty. Let f, g be two contractions on C with coefficient $0 < \alpha < 1$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by* (1.8). *If conditions* (H1)-(H3) *are satisfied, then $x_n \to \tilde{x}$ and $y_n \to \tilde{y}$ as $n \to \infty$, where $\tilde{x} = P_{F(T_1)}f(\tilde{y})$, $\tilde{y} = P_{F(T_2)}g(\tilde{x})$, which solves HOP* (1.6).

Proof First we show that $\{x_n\}$ and $\{y_n\}$ are bounded. Indeed, taking $(p,q) \in F(T_1) \times F(T_2)$, it follows that

$$\begin{aligned} d(x_{n+1},p) + d(y_{n+1},q) &= d\big(\alpha_n f(T_2 y_n) \oplus (1-\alpha_n)T_1 x_n, p\big) \\ &\quad + d\big(\alpha_n g(T_1 x_n) \oplus (1-\alpha_n)T_2 y_n, q\big) \\ &\leq \alpha_n d\big(f(T_2 y_n), p\big) + (1-\alpha_n)d(T_1 x_n, p) \\ &\quad + \alpha_n d\big(g(T_1 x_n), q\big) + (1-\alpha_n)d(T_2 y_n, q) \\ &\leq \alpha_n d\big(f(T_2 y_n), f(q)\big) + \alpha_n d\big(f(q), p\big) + (1-\alpha_n)d(T_1 x_n, p) \\ &\quad + \alpha_n d\big(g(T_1 x_n), g(p)\big) + \alpha_n d\big(g(p), q\big) + (1-\alpha_n)d(T_2 y_n, q) \\ &\leq \alpha_n \alpha d(y_n, q) + \alpha_n d\big(f(q), p\big) + (1-\alpha_n)d(x_n, p) \\ &\quad + \alpha_n \alpha d(x_n, p) + \alpha_n d\big(g(p), q\big) + (1-\alpha_n)d(y_n, q) \\ &= \big(1 - \alpha_n(1-\alpha)\big)\big[d(x_n,p) + d(y_n,q)\big] \\ &\quad + \alpha_n(1-\alpha)\frac{d(f(q),p) + d(g(p),q)}{1-\alpha} \\ &\leq \max\left\{d(x_n,p) + d(y_n,q), \frac{d(f(q),p) + d(g(p),q)}{1-\alpha}\right\}. \end{aligned}$$

By induction, we can prove that

$$d(x_n,p)+d(y_n,q)\le \max\left\{d(x_0,p)+d(y_0,q),\frac{d(f(q),p)+d(g(p),q)}{1-\alpha}\right\} \tag{3.7}$$

for all $n\in\mathbb{N}$. This implies that $\{x_n\}$ and $\{y_n\}$ are bounded, so are $\{T_1x_n\}$, $\{T_2y_n\}$, $\{f(T_2y_n)\}$ and $\{g(T_1x_n)\}$.

We claim that $d(x_{n+1},x_n)\to 0$ and $d(y_{n+1},y_n)\to 0$. Indeed, we have (for some appropriate constant $M>0$)

$$\begin{aligned}
&d(x_{n+1},x_n)+d(y_{n+1},y_n)\\
&\quad= d\big(\alpha_n f(T_2y_n)\oplus(1-\alpha_n)T_1x_n,\alpha_{n-1}f(T_2y_{n-1})\oplus(1-\alpha_{n-1})T_1x_{n-1}\big)\\
&\qquad+d\big(\alpha_n g(T_1x_n)\oplus(1-\alpha_n)T_1y_n,\alpha_{n-1}g(T_1x_{n-1})\oplus(1-\alpha_{n-1})T_2y_{n-1}\big)\\
&\quad\le d\big(\alpha_n f(T_2y_n)\oplus(1-\alpha_n)T_1x_n,\alpha_n f(T_2y_{n-1})\oplus(1-\alpha_n)T_1x_{n-1}\big)\\
&\qquad+d\big(\alpha_n f(T_2y_{n-1})\oplus(1-\alpha_n)T_1x_{n-1},\alpha_{n-1}f(T_2y_{n-1})\oplus(1-\alpha_{n-1})T_1x_{n-1}\big)\\
&\qquad+d\big(\alpha_n g(T_1x_n)\oplus(1-\alpha_n)T_2y_n,\alpha_n g(T_1x_{n-1})\oplus(1-\alpha_n)T_2y_{n-1}\big)\\
&\qquad+d\big(\alpha_n g(T_1x_{n-1})\oplus(1-\alpha_n)T_2y_{n-1},\alpha_{n-1}g(T_1x_{n-1})\oplus(1-\alpha_{n-1})T_2y_{n-1}\big)\\
&\quad\le \alpha_n d\big(f(T_2y_n),f(T_2y_{n-1})\big)+(1-\alpha_n)d(T_1x_n,T_1x_{n-1})\\
&\qquad+|\alpha_n-\alpha_{n-1}|d\big(f(T_2y_{n-1}),T_1x_{n-1}\big)\\
&\qquad+\alpha_n d\big(g(T_1x_n),g(T_1x_{n-1})\big)+(1-\alpha_n)d(T_2y_n,T_2y_{n-1})\\
&\qquad+|\alpha_n-\alpha_{n-1}|d\big(g(T_1x_{n-1}),T_2y_{n-1}\big)\\
&\quad\le \alpha_n\alpha d(T_2y_n,T_2y_{n-1})+(1-\alpha_n)d(T_1x_n,T_1x_{n-1})+|\alpha_n-\alpha_{n-1}|d\big(f(T_2y_{n-1}),T_1x_{n-1}\big)\\
&\qquad+\alpha_n\alpha d(T_1x_n,T_1x_{n-1})+(1-\alpha_n)d(T_2y_n,T_2y_{n-1})+|\alpha_n-\alpha_{n-1}|d\big(g(T_1x_{n-1}),T_2y_{n-1}\big)\\
&\quad\le \big(1-\alpha_n(1-\alpha)\big)\big[d(x_n,x_{n-1})+d(y_n,y_{n-1})\big]+M|\alpha_n-\alpha_{n-1}|.
\end{aligned}$$

By conditions (H2) and (H3) and Lemma 2.5, we have

$$d(x_{n+1},x_n)+d(y_{n+1},y_n)\to 0, \tag{3.8}$$

and thus $d(x_{n+1},x_n)\to 0$, $d(y_{n+1},y_n)\to 0$.

Consequently, by condition (H1), we have

$$\begin{aligned}
d(x_n,T_1x_n)+d(y_n,T_2y_n) &\le d(x_n,x_{n+1})+d(x_{n+1},T_1x_n)\\
&\quad+d(y_n,y_{n+1})+d(y_{n+1},T_2y_n)\\
&= d(x_n,x_{n+1})+d\big(\alpha_n f(T_2y_n)\oplus(1-\alpha_n)T_1x_n,T_1x_n\big)\\
&\quad+d(y_n,y_{n+1})+d\big(\alpha_n g(T_1x_n)\oplus(1-\alpha_n)T_2y_n,T_2y_n\big)\\
&= d(x_n,x_{n+1})+\alpha_n d\big(f(T_2y_n),T_1x_n\big)\\
&\quad+d(y_n,y_{n+1})+\alpha_n d\big(g(T_1x_n),T_2y_n\big)\\
&\to 0\quad(n\to\infty).
\end{aligned} \tag{3.9}$$

This implies that

$$d(x_n, T_1x_n) \to 0, \qquad d(y_n, T_2y_n) \to 0 \quad (\text{as } n \to \infty). \tag{3.10}$$

Let $\{x_t\}$ and $\{y_t\}$ be two nets in C such that

$$\begin{cases} x_t = tf(T_2y_t) \oplus (1-t)T_1x_t, \\ y_t = tg(T_1x_t) \oplus (1-t)T_2y_t. \end{cases}$$

By Theorem 3.2, we have that $x_t \to \tilde{x}$ and $y_t \to \tilde{y}$ as $t \to 0$ such that $\tilde{x} = P_{F(T_1)}f(\tilde{y})$, $\tilde{y} = P_{F(T_2)}g(\tilde{x})$, which solves the variational inequality (1.6). Now, we claim that

$$\limsup_{n\to\infty}\bigl[\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle + \langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n\tilde{y}}\rangle\bigr] \le 0.$$

From Lemma 2.5, we have

$$\begin{aligned}
&d^2(x_t, x_n) + d^2(y_t, y_n) \\
&\quad = \langle\overrightarrow{x_tx_n}, \overrightarrow{x_tx_n}\rangle + \langle\overrightarrow{y_ty_n}, \overrightarrow{y_ty_n}\rangle \\
&\quad \le t\langle\overrightarrow{f(T_2y_t)x_n}, \overrightarrow{x_tx_n}\rangle + (1-t)\langle\overrightarrow{T_1(x_t)x_n}, \overrightarrow{x_tx_n}\rangle \\
&\qquad + t\langle\overrightarrow{g(T_1x_t)y_n}, \overrightarrow{y_ty_n}\rangle + (1-t)\langle\overrightarrow{T_2(y_t)y_n}, \overrightarrow{y_ty_n}\rangle \\
&\quad = t\langle\overrightarrow{f(T_2y_t)f(\tilde{y})}, \overrightarrow{x_tx_n}\rangle + t\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_tx_n}\rangle + t\langle\overrightarrow{\tilde{x}x_t}, \overrightarrow{x_tx_n}\rangle + t\langle\overrightarrow{x_tx_n}, \overrightarrow{x_tx_n}\rangle \\
&\qquad + t\langle\overrightarrow{g(T_1x_t)g(\tilde{x})}, \overrightarrow{y_ty_n}\rangle + t\langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_ty_n}\rangle + t\langle\overrightarrow{\tilde{y}y_t}, \overrightarrow{y_ty_n}\rangle + t\langle\overrightarrow{y_ty_n}, \overrightarrow{y_ty_n}\rangle \\
&\qquad + (1-t)\langle\overrightarrow{T_1(x_t)T_1(x_n)}, \overrightarrow{x_tx_n}\rangle + (1-t)\langle\overrightarrow{T_1(x_n)x_n}, \overrightarrow{x_tx_n}\rangle \\
&\qquad + (1-t)\langle\overrightarrow{T_2(y_t)T_2(y_n)}, \overrightarrow{y_ty_n}\rangle + (1-t)\langle\overrightarrow{T_2(y_n)y_n}, \overrightarrow{y_ty_n}\rangle \\
&\quad \le t\alpha d(y_t, \tilde{y})d(x_t, x_n) + t\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_tx_n}\rangle + td(\tilde{x}, x_t)d(x_t, x_n) + td^2(x_t, x_n) \\
&\qquad + t\alpha d(x_t, \tilde{x})d(y_t, y_n) + t\langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_ty_n}\rangle + td(\tilde{y}, y_t)d(y_t, y_n) + td^2(y_t, y_n) \\
&\qquad + (1-t)d^2(x_t, x_n) + (1-t)d\bigl(T_1(x_n), x_n\bigr)d(x_tx_n) \\
&\qquad + (1-t)d^2(y_t, y_n) + (1-t)d\bigl(T_2(y_n), y_n\bigr)d(y_ty_n) \\
&\quad \le t\alpha d(y_t, \tilde{y})M + t\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_tx_n}\rangle + td(\tilde{x}, x_t)M + td^2(x_t, x_n) \\
&\qquad + t\alpha d(x_t, \tilde{x})M + t\langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_ty_n}\rangle + td(\tilde{y}, y_t)M + td^2(y_t, y_n) \\
&\qquad + (1-t)d^2(x_t, x_n) + (1-t)d\bigl(T_1(x_n), x_n\bigr)M \\
&\qquad + (1-t)d^2(y_t, y_n) + (1-t)d\bigl(T_2(y_n), y_n\bigr)M \\
&\quad \le \bigl[d^2(x_t, x_n) + d^2(y_t, y_n)\bigr] + tM\alpha\bigl[d(x_t, \tilde{x}) + d(y_t, \tilde{y})\bigr] \\
&\qquad + tM\bigl[d(\tilde{x}, x_t) + d(\tilde{y}, y_t)\bigr] \\
&\qquad + M\bigl[d\bigl(T_1(x_n), x_n\bigr) + d\bigl(T_2(y_n), y_n\bigr)\bigr] + t\bigl[\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_tx_n}\rangle + \langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_ty_n}\rangle\bigr],
\end{aligned}$$

where $M \geq \max\{\sup\{d(x_t, x_n) : t \in (0,1), n \geq 0\}, \sup\{d(y_t, y_n) : t \in (0,1), n \geq 0\}\}$. Simplifying this, we have

$$\begin{aligned}
&\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n x_t}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n y_t}\rangle \\
&\quad \leq M(1+\alpha)\big[d(x_t, \tilde{x}) + d(y_t, \tilde{y})\big] \\
&\qquad + \frac{M}{t}\big[d\big(T_1(x_n), x_n\big) + d\big(T_2(y_n), y_n\big)\big].
\end{aligned} \tag{3.11}$$

Taking the limit as $n \to \infty$ first and then letting $t \to 0$ on both sides of (3.11), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\big[\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n x_t}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n y_t}\rangle\big] \leq 0.$$

Since $x_t \to \tilde{x}$ and $y_t \to \tilde{y}$ as $t \to 0$, by the continuity of a metric d, it follows that

$$\limsup_{t\to 0}\big[\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n x_t}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n y_t}\rangle\big] = \langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n \tilde{x}}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n \tilde{y}}\rangle.$$

This implies that, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $t \in (0,\delta)$, we have

$$\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n \tilde{x}}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n \tilde{y}}\rangle \leq \langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n x_t}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n y_t}\rangle + \epsilon. \tag{3.12}$$

First letting $t \to 0$ and taking limit, and then letting $n \to \infty$ and taking the upper limit on (3.12), we obtain

$$\limsup_{n\to\infty}\big[\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n \tilde{x}}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n \tilde{y}}\rangle\big] \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\limsup_{n\to\infty}\big[\langle \overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_n \tilde{x}}\rangle + \langle \overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_n \tilde{y}}\rangle\big] \leq 0.$$

Finally, we prove that $x_n \to \tilde{x}$ and $y_n \to \tilde{y}$ as $n \to \infty$. Indeed, taking $u_n = \alpha_n \tilde{x} \oplus (1-\alpha_n)T_1 x_n$, $v_n = \alpha_n \tilde{y} \oplus (1-\alpha_n)T_2 y_n$ for any $n \in \mathbb{N}$, it follows from Lemma 2.6(i) that

$$\begin{aligned}
&d^2(x_{n+1}, \tilde{x}) + d^2(y_{n+1}, \tilde{y}) \\
&\quad \leq d^2(u_n, \tilde{x}) + d^2(v_n, \tilde{y}) + 2\langle \overrightarrow{x_{n+1}u_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + 2\langle \overrightarrow{y_{n+1}v_n}, \overrightarrow{y_{n+1}\tilde{y}}\rangle \\
&\quad \leq (1-\alpha_n)^2\big[d^2(x_n, \tilde{x}) + d^2(y_n, \tilde{y})\big] + 2\big[\alpha_n\langle \overrightarrow{f(T_2 y_n)u_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\qquad + (1-\alpha_n)\langle \overrightarrow{T_1(x_n)u_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\big] \\
&\qquad + 2\big[\alpha_n\langle \overrightarrow{g(T_1 x_n)v_n}, \overrightarrow{y_{n+1}\tilde{y}}\rangle + (1-\alpha_n)\langle \overrightarrow{T_2(y_n)v_n}, \overrightarrow{y_{n+1}\tilde{y}}\rangle\big] \\
&\quad \leq (1-\alpha_n)^2\big[d^2(x_n, \tilde{x}) + d^2(y_n, \tilde{y})\big] + 2\big[{\alpha_n}^2\langle \overrightarrow{f(T_2 y_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\qquad + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(T_2 y_n)T_1(x_n)}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\qquad + (1-\alpha_n)\alpha_n\langle \overrightarrow{T_1(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + (1-\alpha_n)^2\langle \overrightarrow{T_1(x_n)T_1(x_n)}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\big] \\
&\qquad + 2\big[{\alpha_n}^2\langle \overrightarrow{g(T_1 x_n)\tilde{y}}, \overrightarrow{y_{n+1}\tilde{y}}\rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{g(T_1 x_n)T_2(y_n)}, \overrightarrow{y_{n+1}\tilde{y}}\rangle
\end{aligned}$$

$$\begin{aligned}
&+ (1-\alpha_n)^2\langle\overrightarrow{T_2(y_n)\tilde{y}}, \overrightarrow{y_{n+1}\tilde{y}}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{T_2(y_n)T_2(y_n)}, \overrightarrow{y_{n+1}\tilde{y}}\rangle\big] \\
&= (1-\alpha_n)^2\big[d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y})\big] + 2\alpha_n\big[\langle\overrightarrow{f(T_2y_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{g(T_1x_n)\tilde{y}}, \overrightarrow{y_{n+1}\tilde{y}}\rangle\big] \\
&= (1-\alpha_n)^2\big[d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y})\big] + 2\alpha_n\big[\langle\overrightarrow{f(T_2y_n)f(\tilde{y})}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \langle\overrightarrow{g(T_1x_n)g(\tilde{x})}, \overrightarrow{y_{n+1}\tilde{y}}\rangle + \langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_{n+1}\tilde{y}}\rangle\big] \\
&\le (1-\alpha_n)^2\big[d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y})\big] + 2\alpha_n\alpha\big[d(y_n,\tilde{y})d(x_{n+1},\tilde{x}) + d(x_n,\tilde{x})d(y_{n+1},\tilde{y})\big] \\
&\quad + 2\alpha_n\big[\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_{n+1}\tilde{y}}\rangle\big] \\
&\le (1-\alpha_n)^2\big[d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y})\big] \\
&\quad + \alpha_n\alpha\big[d^2(y_n,\tilde{y}) + d^2(x_{n+1},\tilde{x}) + d^2(x_n,\tilde{x}) + d^2(y_{n+1},\tilde{y})\big] \\
&\quad + 2\alpha_n\big[\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_{n+1}\tilde{y}}\rangle\big],
\end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned}
&d^2(x_{n+1},\tilde{x}) + d^2(y_{n+1},\tilde{y}) \\
&\quad\le \frac{1-(2-\alpha)\alpha_n + \alpha_n^2}{1-\alpha\alpha_n}\big[d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y})\big] + \frac{2\alpha_n}{1-\alpha\alpha_n}\big[\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_{n+1}\tilde{y}}\rangle\big] \\
&\quad\le \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\big[d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y})\big] \\
&\qquad + \frac{2\alpha_n}{1-\alpha\alpha_n}\big[\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_{n+1}\tilde{y}}\rangle\big] + \frac{\alpha_n^2}{1-\alpha}M,
\end{aligned} \tag{3.14}$$

where $M > \sup_{n\ge 0}[d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y})]$. Thus,

$$d^2(x_{n+1},\tilde{x}) + d^2(y_{n+1},\tilde{y}) \le \big(1-\alpha_n'\big)\big[d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y})\big] + \alpha_n'\beta_n', \tag{3.15}$$

where

$$\alpha_n' = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} \quad \text{and} \quad \beta_n' = \frac{(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)^2}M + \frac{1}{1-\alpha}\big[\langle\overrightarrow{f(\tilde{y})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{g(\tilde{x})\tilde{y}}, \overrightarrow{y_{n+1}\tilde{y}}\rangle\big].$$

Applying Lemma 2.6, we have $d^2(x_n,\tilde{x}) + d^2(y_n,\tilde{y}) \to 0$. Hence $x_n \to \tilde{x}$ and $y_n \to \tilde{y}$ as $n \to \infty$. This completes the proof of Theorem 3.3. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors participated in this article's design and coordination. And they read and approved the final manuscript.

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